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THE PSYCHOLOGY OF PROBLEM SOLVING

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(Continued)

THE OVERVALUATION OF VERBAL PROBLEMS

One reason for the great value attached to solving these verbal problems is a confusion of their value as training with their value as tests, and a misunderstanding of what they test. The ability to organize a set of facts in an equation or set of equations such that solving will produce the desired answer is very closely correlated with general intelligence of the scholarly type. The pupils who can do it well rank high in intellect and scholarship. So it is natural to infer that doing it creates and improves the ability. But this inference may be false or at least much exaggerated. Ability in supplying the missing words in sentences is also an excellent test of general intelligence. But the ability certainly has not been created or improved by supplying missing words, since that form of mental gymnastics has not been experienced by pupils save as a feature of psychological tests! The close correlation between ability in solving verbal problems and general ability is perhaps sufficiently accounted for by the fact that the task involves two abilities, each of which is closely related to general ability, namely ability in algebraic computation and ability in paragraph reading. Given a sufficient ability in algebraic computation and in paragraph reading, and pupils might conceivably solve a novel problem almost as well after two hours training in problem solving as after two hundred. Training of course improves their ability to solve the special sorts of problems they practice with, but the value of this depends largely on the genuineness and usefulness of the particular problems used.

Certain students of the teaching of algebra would agree with all this, but insist that the value of the verbal problems as train-

ing in the exact and adequate reading of paragraphs was sufficient to justify the high value attached to them.

This could conceivably be true. Solving a thousand verbal problems certainly has whatever educative value belongs to reading with great care a thousand short paragraphs and doing the thousand relevant computations. It has, indeed, the additional value that belongs to organizing the facts thus carefully read into equational forms such as will give the desired answers. The reading matter of these thousand short paragraphs is, however, so little in amount and so specialized in its nature that the training given by it seems insufficient to justify the high opinion of verbal problems or the time devoted to solving them.

The Use of Problems to Show the Need for a Certain Procedure and to Aid in Mastering It as Well as to Test and Improve the Ability to Apply the Procedure

Other things being equal, it is better for pupils to feel some need for a procedure and purpose in learning it before they learn it. They are then more likely to understand it and much more likely to care about learning it.* Thus writing a real letter is now the beginning rather than the end of the lessons about "Dear Sir" and "Yours truly"; problems about the total cost of several toys or Christmas presents are the beginning rather than the end of the lessons on "carrying" in addition. "Why do we open the draughts of a stove to make the fire burn?" and, "What do we mix with the gasoline in an automobile?" are questions that introduce rather than follow the study of oxygen.

Thus in algebra problems about the average temperature of a series of days varying above and below 0, or about the total of certain credits and penalties in a rating may be excellent features in the introduction to the addition of negative numbers. Problems like the following may be useful as parts of an introduction to "— divided by — gives +."*

*We do not here discuss this general educational axiom because probably it will be acceptable as stated. The whole matter of pupils' purposes in learning, and the special doctrine of "first the need, then the technique," has received a classic general treatment at the hands of Dewey. The case with algebra is much the same as with arithmetic, on which the reader may consult Chapter XIV of "The Psychology of Arithmetic" (Thorndike, '22).

Four boys are rated for strength in comparison with the average for their age.

Arthur is	20
John is	12
Fred is	4
Bert is	8

Supply the missing numbers:

Arthur is........times as far below the average as John.
Arthur is........times as far below the average as Fred.
Arthur is........times as far below the average as Bert.
John is........times as far below the average as Fred.

Other things are not always equal. There may be no vital, engaging problems to use as introductory material. For examples, there is not, to my knowledge, any problem that is vital and engaging to the average high-school pupil by which to introduce the general symbolism of fractional exponents. Such problems, though in existence, may use up more time than can be spared. There is, for instance, a problem almost perfectly adapted to arouse the need for knowledge of the laws of signs in multiplication, namely, the problem of measuring resemblance between a pair of measures both of which are divergences from a type or average. But it takes so long to teach the meaning of "resemblance" in such cases that probably the game is not worth the candle. The procedure may be so intrinsically valuable and interesting that mere contact with it will quickly inspire a desirable purpose and activity. For example, gifted pupils will probably learn that $\sqrt{a}\sqrt{a} = a$ as readily by straightforward consideration of $(\sqrt{4}\sqrt{4})$, $(\sqrt{9}\sqrt{9})$, $(\sqrt{16}\sqrt{16})$, $(\sqrt{2}\sqrt{2})$, and $(\sqrt{3}\sqrt{3})$, as by any introductory problem to display the need of knowing that the square root of any number times the square root of the same number equals the number.

Criteria in Selecting Problems

In this section, as in the previous one, we are concerned not alone with problems where an equation or equations are used to discover certain particular quantities relating to one particular state of affairs, but with problem material in general.

Solving problems in school is for the sake of problem solving in life. Other things being equal, problems where the situation is real are better than problems where it is described in words.

*The illustrations here are not problems where organization in the equational form is necessary. What is said about problems in this section, indeed concerns all problems of types I and II, not merely the II-B-b problems.

Other things being equal, problems which might really occur in a sane and reasonable life are better than bogus problems and mere puzzles. Other things being equal, problems which give desirable training in framing equations from the realities or the verbal statements are better than problems which give training chiefly in solving the equations when framed. The latter training can be got easily by itself.

As was suggested in an earlier article, a better selection of problems will probably be secured if, instead of searching for problems which conveniently apply to fractional equations, problems to apply simultaneous linear equations, and so on, we search for problems which are intrinsically worth learning to solve by algebraic methods.

If it happens that there are no genuine, important problems calling for the framing and solution of a certain technique, say simultaneous quadratics, we may simply leave that technique without application to verbal problems or we may frankly provide problems that make no false pretenses at reality as in "I am thinking of two numbers, . . . etc."

This case of simultaneous quadratics is a good one to illustrate the two points of view contrasted here. The older view, in order to have applications of simultaneous quadratics, fabricated extraordinary tasks depending on insane curiosity to know the dimensions of a field which, when altered in various ways, gives fields of certain areas, and the like. The newer view selects first the case of determining the constants in a quadratic equation from knowledge of the (x) (y) values of certain points on the curve. The ability to do this is not of great "social utility" to many of the individuals who study ninth or tenth grade mathematics, and might well be left for those who specialize further in mathematics or science. It is, however, a genuine problem. The next choice of the newer view would probably be the solution of ". . . and . . . are two curves. Do they intersect? If so, at what points?" This again is not a question which life puts to many persons, but it is one that a sane person need not be ashamed to ask. If genuine applications of the technique of simultaneous equations are beyond the abilities and interests of high school pupils, we may leave it without application until the abilities and interests are available.

Problems as Tests

Genuine application in the real world should be demanded in problems which are used for training, and is preferable in problems given simply to test the ability to organize a set of statements into equations to answer a question. It is preferable in the later case because, human nature being what it is, teachers will be prone to train for the test. The nature of the examinations used has always influenced the nature of the instruction and probably always will. Except for this, we might permit as algebraic "originals" in tests, the problems about consecutive digits, hands of a watch, numerators and denominators defined by their sums, differences, products and quotients in fantastic ways, and the like, which we exclude from mathematical *training* if better material can be obtained.

Actual and Described Situations

We have noted in an earlier article that the ability to manage a problem as encountered in reality, and the ability to manage the same problem as it is described in words in an algebra book need not be identical. Success with the former is consistent with failure with the latter, and *vice versa*.

The worst discrepancies are when, on the one hand, a state of affairs which would be very clear and comprehensible to a person experiencing it, is beclouded and confused by words, and when, on the other hand, the pupil learns to obtain correct solutions by response to verbal cues, the lack of which would cause the real situation, when encountered, to baffle him.

As an example of the first, consider this problem:

"A man is paying for a $300 piano at the rate of $10.00 per month with interest at 6%. Each month he pays *the total interest which has accrued on that month's payment.* How much money, including principal and interest, will he have paid when he has freed himself from the debt?" In reality the man probably would be told that he had to pay $10.05 the first month; $10.10 the second month; $10.15 the third month, etc., and would easily see the progression. The difficulty with the problem in schools lies chiefly in understanding what "each month he pays the total interest which had accrued on that month's payment" means, and in the confusing use of "including principal and interest" and "debt."

It seems wiser to give more attention to providing real situations for the application of algebra. For example, it seems wise for pupils to draw a straight line and another cutting it, and find the size of all four angles by measuring one of them, as well as to solve problems in words about supplementary angles. There is not only a greater surety that the pupils are being prepared to respond effectively to situations which life will actually offer, and an insurance against the danger of unsuitable linguistic demands, but also often an increase of interest in and respect for algebra.

Like almost everything in teaching, we have to add the clause "other things being equal" to this recommendation. Too much time must not be spent in drawing, measuring, weighing and the like. Also, the "real" situation will often be a map already drawn, a table of values already measured, a set of observations already made. Also the genuine problems to which algebra applies are, as compared with those of arithmetic, more often prophetic, foretelling what will happen if certain conditions are fulfilled or what to do in order to bring certain results to pass. The genuine task is, in many of these prophetic problems, precisely to understand a verbal description. Such, for example, are problems about mixtures and alloys, about the amount of d needed to have its proportion to c the same as b's to a, and about the drawing of rectangles of specified proportions and total area.

Isolated and Grouped Problems

Problems grouped by their relation to some aspect of science, industry, business and home life, as *Falling bodies,* or *Alloys,* or *Sliding scales for wages* or *Dietaries* have certain advantages. (1) The situations dealt with are more likely to be understood; (2) things are put together in the pupil's mind that belong together in logic or in reality or in both; (3) the data needed for all the problems can be given once for all, so that in each problem the pupil has to select the facts needed to answer it as well as to arrange them in suitable equations.

The isolated problem is indeed disappearing from arithmetic except in special exercises for particular purposes of tests, reviews, and training in alertness and adaptation. It appears

that by the exercise of enough care and ingenuity, problems in arithmetic can be grouped in this way with no loss to the purely arithmetical training that is given. The same tendency is operating in algebra, and much good may be expected from it.

Problems Requiring the Selection of Data

The third advantage noted above as characteristic of grouped problems is of special importance, as was noted in an earlier article.

The custom is firmly fixed of giving in a problem only the facts needed to solve it, so that "there are as many distinct statements as there are unknown numbers"; and the pupil is taught to "represent one of the unknown numbers by a letter; then, using all but one of the statements, represent the other unknowns in terms of that same letter. Using the remaining statement, form an equation." Yet it seems unjustifiable. The time and thought now spent by pupils on intricate fabrications whose like they will never see again, might much better be spent in such selective tasks as are genuine and instructive.

Problems Requiring the Discovery of Data

A further step is worth consideration, namely, that of giving problems some of the data for which are lacking and must be supplied by the pupil's search. We do this to a slight extent by not including in the statement of a problem such needed facts that 1 foot = 12 inches, or that a square had four equal sides. Is it desirable to require the pupil to find in his memory or in tables at the end of his textbook on algebra or in other reference books or from observation and measurement such facts as the inter-equivalences of inches and centimeters, the weight of a cubic foot of water, the capacity of a 4 ounce bottle, the length and width of his classroom or the area of Ohio?

There are obvious inconveniences in doing this, but there is the advantage of making problem solving in school one degree more like problem solving in science or industry or business. We might at least go as far as to assign a score of problems each with the question, "What further fact or facts must you have in order to solve this problem?", and distribute the work of discovering these facts among the pupils. The lesson that one must often supplement the facts given by the situation itself

by further investigations would then be taught to all, at no great cost of time.

Such searching is, of course, not algebra; neither is the understanding of statements about rates, speeds, investments and yields algebra. The algebra begins when statements understood are to be translated into algebraic symbols. Having already far overstepped that line in the customary work with verbal problems, we may go farther with no inconsistency.

Problems Requiring General Solutions

The most objectionable feature of problem solving in algebra today to a psychologist is the predominance of problems seeking a particular fact about some particular state of affairs—the relative neglect of problems which seek the general relation between variations in one thing and variations in something related to it.

The main service of algebra, as the psychologist sees it, is to teach pupils that we can frame general rules for operating so as to secure the answer to *any* problem of a certain sort, and express these rules with admirable brevity and clearness by literal symbolism. We take great pains to teach the pupil that pq means the product of whatever number we let p equal and what ever number we let q equal; and that if p and q equal any two numbers, the first number times the product of the two equals p^2q, and other similar facts. Then, in problems, the p's and q's or x's and y's in nine cases out of ten, mean something as unlike "any number" as could possibly be. We build up habits of computing with literal numbers and then, in problems, make almost no use thereof, reverting to an arithmetic plus negative numerals with a written x in place of the mental "What I am trying to find." Small wonder that the pupil often thinks of his algebraic computations as a mere game that one plays with a, b, c, d, $+$, $-$, $\times$, $\div$, and $()$. If, after a few exercises in the use of letters to mean "any number of so and so," and a few exercises in reading and framing formulae, we have him do nothing with literal symbols but play computing games, why should he think otherwise?

Why should we blow hot and cold in this way, asserting that algebra teaches us what is true of any number and then, in

problems, making its linear equations true of only a single number, and its quadratics of only two? Should we not alter many of our IIBb problems into the IIBa form, requiring the pupil to frame the general equations or formulae to solve any problem of that sort, and to obtain any particular answer by evaluating? For example, compare the two tasks I and II below:

I. A man has a lawn 40 ft. long and 30 ft. wide. How wide a strip must he mow beginning at the outside edge in order to mow half of it?

II. 1. A man has a rectangular lawn. Make a formula to state how wide a strip he must mow beginning at the outside edge in order to mow half of it. Let l and w equal the length and width of the lawn in feet.

Let s equal the width of the strip in feet.

2. Find s if $l = 40$ and $w = 30$.
3. Find s if $l = 100$ and $w = 20$.
4. Find s if $l = 80$ and $w = 40$.
5. Find s if $l = 80$ and $w = 60$.

It seems reasonable to progress from problems of the I type to problems of the II type just as we progress from numbers to letters, and from such facts as $2 \times 2 = 2^2$, or $3 \times 3 = 3^2$ to such facts as $a \times a = a^2$, or $a(b + c) = ab + ac$.

Amongst problems requiring a general solution in terms of a literal formula, special importance attaches to problems of direct and inverse proportion, problems where one number varies as the square or square root of another, and other problems involving linear, hyperbolic and parabolic relations.

Problems of Puzzle and Mystery

The earliest problems of algebra were problems of puzzle and mystery, such as Ahmes' "A hau, its seventh, it equals 18," or the finding of the age of Diophantus from his epitaph. "Diophantus passed one-sixth of his life in childhood, one-twelfth in youth and one-seventh more as a bachelor; five years after his marriage a son was born, who died four years before his father did at half the age at which his father died."

Such problems make an appeal to certain human interests. Some pupils doubtless prefer them to straight-forward uses of algebra in answering questions of ordinary life. The human tendency to enjoy doing what we can do well, and especially

what we can do better than others can, is often stronger than the tendency to enjoy doing what we know will profit us. Some of these problems are also arranged as strong stimuli to thought for thought's sake. By introducing an element of humor they may relieve the general tension of algebraic work, as is at times desirable. They are much more appropriate in algebra for the selected group of superior pupils who continue to high school than they are in arithmetic for all children. On the whole, however, the ordinary applications of algebra to science, industry, business and the home will give better training to the general run of high school freshmen and will inspire greater liking and respect for mathematics than will these appeals to the interest in puzzles and mystery.

One of the best forms of appeals to the puzzle interest is by abstract problems such as: "When will a^2 be less than a?" "When will l divided by a be greater than a?" "State a condition such that $\frac{a}{b}$ will equal $\frac{b}{a}$?" "State a condition such that abc will equal a?"

One of the best forms of appeal to the interest in mystery is to have pupils frame formulas* for such mysteries as: "Think of any number and I will tell you what it is. Think of the number. Add 3 to it. Multiply the result by 7. Subtract 20. Tell me the result. The number you thought was . . . (This result diminished by 1 and then divided by 7)." They may also make up such mysteries for the class, score being kept of the time required for pupils to find the formula for the mystery and penalties being attached to devising a "mystery" that doesn't work. $(a+b)(a-b)=a^2-b^2$ may be taught as a mystery for quickly computing products like 2998×3002, or 4980×5020. The formula for the sum of an arithmetic progression may be taught as a mystery for computing the sums of such series, either complete, or in the form "All the numbers from . . . to . . except . . . and . . ." There are, however, better motives to use for mastering

$$S=\frac{n}{2}\,[2a+(n-l)d]$$

As has been so often insisted, the cardinal sin in connection

*Representative problems of this sort will be found in Nunn, '13, p. 87 f.

with problems of puzzle and mystery is their decoration with a description of conditions and events in nature which makes the pretence that the problem is genuine when it is not; and so confuses and debauches the pupil's ideas of the uses of algebra. If they are presented in their true light and if the pupils have the option of solving them or solving problems of genuine application,—they can at the worst, do very little harm.

The Election of Problems by Students

Many of the difficulties of teaching in the case of problems are greatly lessened by arranging to have each choose a certain number of problems to solve from a list which contains, say, five times as many as any one pupil is to solve. We have just noted the value of election between "useful" and "puzzle" problems, if the latter are presented at all. We have noted previously that problems drawn from physics may be of very different value to pupils who are studying general science and to pupils who are not. Within the latter group, we might also differentiate between those who happen to be ignorant of science, and those who are so by their own volition. Boys and girls may well differ in their choices, though probably not so much as some theorists would expect. It may be desirable to permit and even encourage some pupils to choose the easier problems.* The provision of five times the number of verbal problems now given in standard textbooks would add perhaps two cents to the cost of production.

Summary

It is a worthy aim to teach pupils to organize the facts of important situations requiring numerical responses into equation form and to solve their equations. It is also worth while for pupils to learn that any quantitative question, no matter how elaborate and intricate, can be so expressed, provided adequate data are given.

Even if the educative value of this work is improved by such modifications as have been suggested in this chapter, it will still be, on the whole, less important than the framing of general equations or formulas for solving *any* problem of a certain kind. Learning to let x or q equal the unknown and to express data

*The instructions may be "Do the ten hardest ones that you think you can do."

in terms of their relations to it is a useful lesson, but learning to express a set of relations in generalized form is a more useful one, and, so far as psychology can prophesy, one more likely to transfer its improvement to other abilities. It is when the verbal problems of algebra advance beyond arithmetical problems in the same way that algebraic computation advances beyond arithmetical computation that they perform their chief educational service.